Problem 1.38

Express the unit vectors $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ in terms of $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ (that is, derive Eq. 1.64). Check your answers several ways ($\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} \stackrel{?}{=} 1$, $\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\phi}} \stackrel{?}{=} 0$, $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} \stackrel{?}{=} \hat{\boldsymbol{\phi}}$, ...). Also work out the inverse formulas, giving $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ in terms of $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ (and θ, ϕ).

Solution

Eq. 1.62 gives the formulas to switch from Cartesian coordinates (x, y, z) into spherical coordinates (r, ϕ, θ), θ being the angle from the polar axis.

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \tag{1.62}$$

The position vector from the origin $(0, 0, 0)$ to the point (x, y, z) is written as

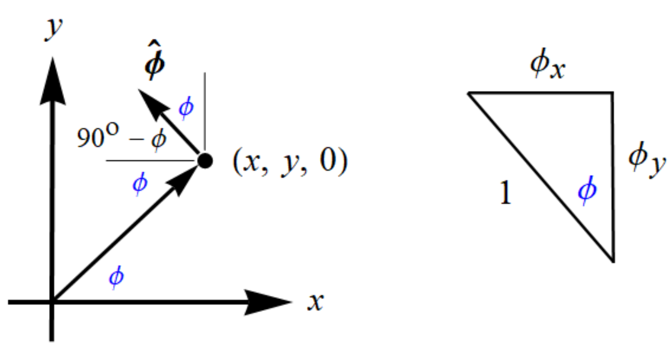
$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

$$r\hat{\mathbf{r}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}.$$

Divide both sides by r to get the radial unit vector.

$$\begin{aligned} \hat{\mathbf{r}} &= \frac{x}{r}\hat{\mathbf{x}} + \frac{y}{r}\hat{\mathbf{y}} + \frac{z}{r}\hat{\mathbf{z}} \\ &= \frac{r\sin\theta\cos\phi}{r}\hat{\mathbf{x}} + \frac{r\sin\theta\sin\phi}{r}\hat{\mathbf{y}} + \frac{r\cos\theta}{r}\hat{\mathbf{z}} \\ &= \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \end{aligned}$$

In order to get a formula for $\hat{\boldsymbol{\phi}}$, consider a point in the xy-plane for simplicity and use trigonometry. The triangle on the right is of the magnitude of $\hat{\boldsymbol{\phi}}$ and the magnitudes of the components along the x- and y-axes.



The x-component of $\hat{\boldsymbol{\phi}}$ is $-\sin\phi$, and the y-component of $\hat{\boldsymbol{\phi}}$ is $\cos\phi$:

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}.$$

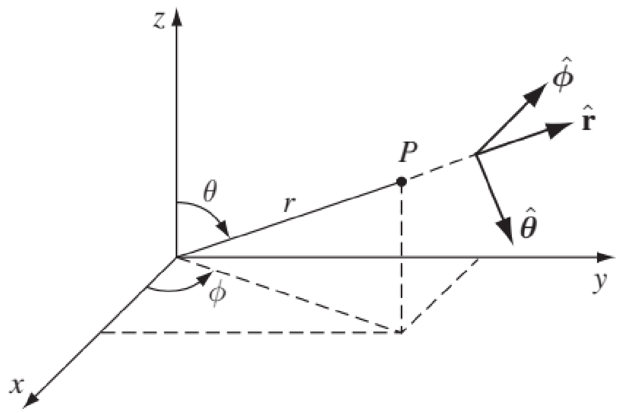


Fig. 1.36

By the right-hand corkscrew rule, the remaining unit vector $\hat{\boldsymbol{\theta}}$ is given by

$$\begin{aligned}
\hat{\boldsymbol{\theta}} &= \hat{\boldsymbol{\phi}} \times \hat{\mathbf{r}} \\
&= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{vmatrix} \\
&= \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} + (-\sin\theta\sin^2\phi - \sin\theta\cos^2\phi)\hat{\mathbf{z}} \\
&= \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}.
\end{aligned}$$

To summarize,

$$\begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \end{cases}.$$

In order to get the formulas for $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, write this as a matrix equation.

$$\begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{\theta}} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -\sin\phi & \cos\phi & 0 \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{bmatrix}$$

Consequently,

$$\begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -\sin\phi & \cos\phi & 0 \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \end{bmatrix}^{-1} \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{\theta}} \end{bmatrix}.$$

Find the inverse of the matrix.

$$\left[\begin{array}{ccc|ccc}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta & 1 & 0 & 0\\ -\sin\phi & \cos\phi & 0 & 0 & 1 & 0\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta & 0 & 0 & 1\end{array}\right] \sim \left[\begin{array}{ccc|ccc}1 & \frac{\sin\phi}{\cos\phi} & \frac{\cos\theta}{\sin\theta\cos\phi} & \frac{1}{\sin\theta\cos\phi} & 0 & 0\\ -\sin\phi & \cos\phi & 0 & 0 & 1 & 0\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta & 0 & 0 & 1\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc}1 & \frac{\sin\phi}{\cos\phi} & \frac{\cos\theta}{\sin\theta\cos\phi} & \frac{1}{\sin\theta\cos\phi} & 0 & 0\\ 0 & \frac{\sin^2\phi}{\cos\phi}+\cos\phi & \frac{\cos\theta\sin\phi}{\sin\theta\cos\phi} & \frac{\sin\phi}{\sin\theta\cos\phi} & 1 & 0\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta & 0 & 0 & 1\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc}1 & \frac{\sin\phi}{\cos\phi} & \frac{\cos\theta}{\sin\theta\cos\phi} & \frac{1}{\sin\theta\cos\phi} & 0 & 0\\ 0 & \frac{\sin^2\phi}{\cos\phi}+\cos\phi & \frac{\cos\theta\sin\phi}{\sin\theta\cos\phi} & \frac{\sin\phi}{\sin\theta\cos\phi} & 1 & 0\\ 0 & \cos\theta\sin\phi-\frac{\sin\phi\cos\phi\cos\theta}{\cos\phi} & -\sin\theta-\frac{\cos^2\theta\cos\phi}{\sin\theta\cos\phi} & -\frac{\cos\theta\cos\phi}{\sin\theta\cos\phi} & 0 & 1\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc}1 & \frac{\sin\phi}{\cos\phi} & \frac{\cos\theta}{\sin\theta\cos\phi} & \frac{1}{\sin\theta\cos\phi} & 0 & 0\\ 0 & \frac{1}{\cos\phi} & \frac{\cos\theta\sin\phi}{\sin\theta\cos\phi} & \frac{\sin\phi}{\sin\theta\cos\phi} & 1 & 0\\ 0 & 0 & -\frac{1}{\sin\theta} & -\frac{\cos\theta}{\sin\theta} & 0 & 1\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc}1 & \frac{\sin\phi}{\cos\phi} & \frac{\cos\theta}{\sin\theta\cos\phi} & \frac{1}{\sin\theta\cos\phi} & 0 & 0\\ 0 & \frac{1}{\cos\phi} & \frac{\cos\theta\sin\phi}{\sin\theta\cos\phi} & \frac{\sin\phi}{\sin\theta\cos\phi} & 1 & 0\\ 0 & 0 & 1 & \cos\theta & 0 & -\sin\theta\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc}1 & \frac{\sin\phi}{\cos\phi} & \frac{\cos\theta}{\sin\theta\cos\phi} & \frac{1}{\sin\theta\cos\phi} & 0 & 0\\ 0 & 1 & \frac{\cos\theta\sin\phi}{\sin\theta} & \frac{\sin\phi}{\sin\theta} & \cos\phi & 0\\ 0 & 0 & 1 & \cos\theta & 0 & -\sin\theta\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc}1 & \frac{\sin\phi}{\cos\phi} & \frac{\cos\theta}{\sin\theta\cos\phi} & \frac{1}{\sin\theta\cos\phi} & 0 & 0\\ 0 & 1 & 0 & \frac{\sin\phi}{\sin\theta}-\frac{\cos^2\theta\sin\phi}{\sin\theta} & \cos\phi & \cos\theta\sin\phi\\ 0 & 0 & 1 & \cos\theta & 0 & -\sin\theta\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc}1 & \frac{\sin\phi}{\cos\phi} & 0 & \frac{1}{\sin\theta\cos\phi}-\frac{\cos^2\theta}{\sin\theta\cos\phi} & 0 & \frac{\cos\theta}{\cos\phi}\\ 0 & 1 & 0 & \sin\phi\sin\theta & \cos\phi & \cos\theta\sin\phi\\ 0 & 0 & 1 & \cos\theta & 0 & -\sin\theta\end{array}\right]$$

Continue the simplification.

$$\left[\begin{array}{ccc|ccc} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta & 1 & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 & 1 & 0 \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & \frac{\sin\phi}{\cos\phi} & 0 & \frac{\sin\theta}{\cos\phi} & 0 & \frac{\cos\theta}{\cos\phi} \\ 0 & 1 & 0 & \sin\phi\sin\theta & \cos\phi & \cos\theta\sin\phi \\ 0 & 0 & 1 & \cos\theta & 0 & -\sin\theta \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{\sin\theta}{\cos\phi} - \frac{\sin^2\phi\sin\theta}{\cos\phi} & -\sin\phi & \frac{\cos\theta}{\cos\phi} - \frac{\cos\theta\sin^2\phi}{\cos\phi} \\ 0 & 1 & 0 & \sin\phi\sin\theta & \cos\phi & \cos\theta\sin\phi \\ 0 & 0 & 1 & \cos\theta & 0 & -\sin\theta \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \cos\phi\sin\theta & -\sin\phi & \cos\phi\cos\theta \\ 0 & 1 & 0 & \sin\phi\sin\theta & \cos\phi & \cos\theta\sin\phi \\ 0 & 0 & 1 & \cos\theta & 0 & -\sin\theta \end{array}\right]$$

The inverse matrix is then

$$\begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -\sin\phi & \cos\phi & 0 \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \end{bmatrix}^{-1} = \begin{bmatrix} \cos\phi\sin\theta & -\sin\phi & \cos\phi\cos\theta \\ \sin\phi\sin\theta & \cos\phi & \cos\theta\sin\phi \\ \cos\theta & 0 & -\sin\theta \end{bmatrix},$$

which means

$$\begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -\sin\phi & \cos\phi & 0 \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \end{bmatrix}^{-1} \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{\theta}} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi\sin\theta & -\sin\phi & \cos\phi\cos\theta \\ \sin\phi\sin\theta & \cos\phi & \cos\theta\sin\phi \\ \cos\theta & 0 & -\sin\theta \end{bmatrix} \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{\theta}} \end{bmatrix}.$$

Therefore,

$$\begin{cases} \hat{\mathbf{x}} = \cos\phi\sin\theta\,\hat{\mathbf{r}} - \sin\phi\,\hat{\boldsymbol{\phi}} + \cos\phi\cos\theta\,\hat{\boldsymbol{\theta}} \\ \hat{\mathbf{y}} = \sin\phi\sin\theta\,\hat{\mathbf{r}} + \cos\phi\,\hat{\boldsymbol{\phi}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}.$$